Solution to Problem 5) From the defining equation $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \mathrm{d}t$ it is clear that the conjugate of $\Gamma(z)$ is $\Gamma(z^*)$. Of course, the above integral is valid only for the right-half of the complex z-plane, where $\mathrm{Re}(z) > 0$. However, analytic continuation of $\Gamma(z)$ to the left-half of the z-plane preserves this property, so that, in general, $\Gamma^*(z) = \Gamma(z^*)$. The only other formulas needed to prove the asserted identities are the functional relation $\Gamma(z+1) = z\Gamma(z)$, and Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$. We will then have

a) $$|\Gamma(\mathrm{i}y)|^2 = \Gamma(\mathrm{i}y)\Gamma^*(\mathrm{i}y) = \Gamma(\mathrm{i}y)\Gamma(-\mathrm{i}y) = \Gamma(\mathrm{i}y)\,(-\mathrm{i}y)\Gamma(-\mathrm{i}y)/(-\mathrm{i}y)$$

$$= \frac{\Gamma(\mathrm{i}y)\Gamma(1-\mathrm{i}y)}{(-\mathrm{i}y)} = \frac{\pi}{\sin(\mathrm{i}\pi y)(-\mathrm{i}y)} = \frac{2\mathrm{i}\pi}{(e^{-\pi y} - e^{\pi y})(-\mathrm{i}y)} = \frac{\pi}{y\sinh(\pi y)}.$$

b) $$|\Gamma(½ + \mathrm{i}y)|^2 = \Gamma(½ + \mathrm{i}y)\Gamma^*(½ + \mathrm{i}y) = \Gamma(½ + \mathrm{i}y)\Gamma(½ - \mathrm{i}y)$$

$$= \Gamma(½ + \mathrm{i}y)\Gamma(1 - ½ - \mathrm{i}y) = \pi/\sin[\pi(½ + \mathrm{i}y)]$$

$$= \frac{2\mathrm{i}\pi}{\exp[\mathrm{i}\pi(½ + \mathrm{i}y)] - \exp[-\mathrm{i}\pi(½ + \mathrm{i}y)]} = \frac{2\pi}{\exp(-\pi y) + \exp(\pi y)} = \frac{\pi}{\cosh(\pi y)}.$$

c) $$\Gamma(1 + \mathrm{i}y)\Gamma(1 - \mathrm{i}y) = (\mathrm{i}y)\Gamma(\mathrm{i}y)\Gamma(1 - \mathrm{i}y) = \mathrm{i}\pi y/\sin(\mathrm{i}\pi y)$$

$$= 2\mathrm{i}^2\pi y/(e^{-\pi y} - e^{\pi y}) = \pi y/\sinh(\pi y).$$
